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MAPPING CYLINDERS OF HILBERT CUBE FACTORS II. THE RELATIVE CASE

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A concept of relative or reduced mapping cylinders is introduced, and it is shown that if f is any map between two spaces which have the property that their products with the Hilbert cube are compact Hilbert cube manifolds, then for any closed subset A of the domain of f the relative mapping cylinder of f reduced modulo A also has this property and that the two collapses associated with it generate in a natural way near-homeomorphisms of Hilbert cube manifolds.

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Hilbert cube factor	near-homeomorphism
Hilbert cube manifold factor	relative mapping cylinder

1. Introduction

In [7] it was shown that mapping cylinders of arbitrary maps between Hilbert cube factors (spaces whose Cartesian product with another space is a Hilbert cube) are themselves Hilbert cube factors and moreover that, the natural collapses of these mapping cylinders generate "near-homeomorphisms" when appropriately "stabilized". That work is here extended to "relative" (or "reduced") mapping cylinders of maps between compact spaces whose products with the Hilbert cube are Hilbert cube manifolds. The general motivation for such theorems is the identification and study of Absolute Neighborhood Retracts with this property, and, in fact, the immediate motivation for both [7] and this paper was the work of the author and R.M. Schori on hyperspaces [3, 4, 5, 8, 9] during which it was necessary to develop more resources for identifying Hilbert cube factors and maps which "stabilize" to "near-homeomor-

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phisms". The main result of [7] was used in [10] to establish a sum theorem for Hilbert cube factors, and Chapman has observed that the analogous sum theorem for spaces stabilizing to compact Hilbert cube manifolds follows immediately from that and the present work (see [10]).

2. Definitions and conventions

It is necessary to make precise the notion of stabilization used here. To begin with, the unit interval $[0, 1]$ is denoted by I. The Hilbert cube Q is the countably infinite product $\Pi_{i=1}^{\infty} I_i$ of I with itself ($I_i = I$, $i = 1, 2, \ldots$). To *stabilize a space* X is to take its product $X \times Q$ with the Hilbert cube, $X \times Q$ being the *stabilization of* X; to *stabilize a map* $f : X \to Y$ is to take its product

$$f \times 1_Q : X \times Q \to Y \times Q$$

with the identity mapping of Q, $f \times 1_Q$ being the *stabilization of* f. A map $f : X \to Y$ is a *near-homeomorphism* if it is a uniform limit of homeomorphisms of X onto Y (all homeomorphisms are surjections), and a map *stabilizes to a near-homeomorphism* if its stabilization is a near-homeomorphism.

For any map $f : X \to Y$ between compact metric spaces, the *mapping cylinder* $M(f)$ of f is the quotient space

$$(X \times I \cup Y)/\sim,$$

where $\sim$ is the equivalence relation generated by the condition $(x,0) \sim f(x)$ for each $x \in X$. Equivalence classes (points) in $M(f)$ are denoted by brackets, $[x, t]$ and $[y]$, but no distinction is made between Y and its natural image in $M(f)$, which is the *base* of $M(f)$; thus

$$[y] = y \in Y \subset M(f).$$

The quotient map is denoted by

$$\pi(f) : (X \times I \cup Y) \to M(f),$$

and the *collapse*

$$c(f) : M(f) \to Y$$

of $M(f)$ to its base is the natural retraction defined by

$$c(f)([x, t]) = f(x), \qquad c(f) \mid Y = 1_Y.$$

If A is a closed subset of X, the *relative mapping cylinder* $M(f, A)$ *of* f *reduced modulo* A is the quotient space of $M(f)$ under the equivalence relation $\equiv$ generated by the condition

$$[a, t] \equiv f(a) \quad \text{for each } a \in A.$$

Members of $M(f, A)$ are also denoted by square brackets, and the quotient map is denoted by

$$\pi(f, A) : M(f) \to M(f, A);$$

thus

$$\pi(f, A)([x, t]) = [x, t],$$

but there is no possibility of confusion. Again, $Y = \pi(f, A)(Y)$ is identified with the base of $M(f, A)$. The collapse $c(f, A) : M(f, A) \to Y$ of $M(f, A)$ is $c(f)\, \pi(f, A)^{-1}$.

Finally, *simplicial complexes* are taken to be spaces (polyhedra) with explicit triangulations, finite simplicial complexes being then the compact ones, i.e., those with finitely many simplices, and L is a subcomplex of K if it is a subset of K and each simplex of its triangulation is one of that of K. A *finite simplicial pair* (K, L) is a finite simplicial complex K together with a subcomplex L of K, and (H, K, L) is a finite simplicial *triple* if H is a finite simplicial complex, K is a subcomplex of H, and L is a subcomplex of K.

3. Lemmas

In the terminology of stabilizations [7, Theorem 2] becomes the following (Hilbert cube factors stabilize to Hilbert cubes[7], see also [6, Theorem 6.2]).

Lemma 1 (Collapse Stabilization Theorem). *Let X stabilize to a Hilbert cube and Y stabilize to a compact Hilbert cube manifold. Then the mapping cylinder of any map from X to Y stabilizes to a Hilbert cube manifold and its collapse stabilizes to a near-homeomorphism.* □

The next lemma is essentially a relative version of [7, Corollary 7].

Lemma 2. *If Y stabilizes to a compact Hilbert cube manifold, then for any finite simplicial pair (K, L) and map $f : K \times Q \to Y$, the relative* *mapping cylinder $M(f, L \times Q)$ stabilizes to a Hilbert cube manifold homeomorphic to $Y \times Q$, and its collapse $c(f, L \times Q)$ stabilizes to a near-homeomorphism.*

Proof. That $M(f, L \times Q)$ stabilizes to a Hilbert cube manifold and $c(f, L \times Q)$ stabilizes to a near-homeomorphism follows easily from an inductive use of Lemma 1 on the simplices of K together with the observation that if σ is a simplex with boundary $\partial\sigma$, then $\sigma \times I$ may be viewed as a mapping cylinder M with base $\sigma \times \{0\} \cup (\partial\sigma \times I)$ and collapse c in such a manner that if $p : \sigma \times I \to \sigma$ is the projection, then $pc : M \to \sigma$ is arbitrarily close to p. □

It is convenient to represent reduced mapping cylinders as subsets of the unreduced mapping cylinders. Let $f : X \to Y$ be a map between compact metric spaces, and let $\varphi : X \to I$ be any map. Let

$$Z(\varphi) = Y \cup \{[x, t] \mid t \leqslant \varphi(x)\} \subset M(f),$$

and observe that the function

$$h : M(f, \varphi^{-1}(0)) \to Z(\varphi)$$

given by $h([x, t]) = [x, \varphi(x)t]$ is a homeomorphism such that $c(f)h = c(f, \varphi^{-1}(0))$. Now if $\psi : X \to I$ is another map and $\psi(x) \leqslant \varphi(x)$ for each $x \in X$, let $r_{\varphi\psi} : Z(\varphi) \to Z(\psi)$ be the natural retraction

$$(r_{\varphi\psi}([x, t]) = [x, \min\{t, \psi(x)\}]).$$

If $\varphi(X) = \{1\}$, $r_{\varphi\psi}$ will be denoted by r_ψ.

Lemma 3. *If Y stabilizes to a compact Hilbert cube manifold, then for any finite simplicial triple (H, K, L), any map $f : H \times Q \to Y$, and any two maps $\varphi, \psi : X \to I$ with $\psi(x) \leqslant \varphi(x)$ for each $x \in X = H \times Q$ such that*

$$(K \times Q, L \times Q) = (\psi^{-1}(0), \varphi^{-1}(0)),$$

then the retraction $r_{\varphi\psi}$ stabilizes to a near-homeomorphism.

Proof. It may be assumed without loss of generality that $\varphi(X) \subset [0, \frac{1}{2}]$. Define $g_\varphi : H \times Q \to Z(\varphi)$ and $g_\psi : H \times Q \to Z(\psi)$ by $g_\varphi(x) = [x, \varphi(x)]$ and $g_\psi(x) = [x, \psi(x)]$. There are homeomorphisms $h_\varphi : M(f) \to M(g_\varphi)$ and $h_\psi : M(f) \to M(g_\psi)$ for which the following diagram commutes:

$$\begin{tikzcd}
M(g_\varphi) \arrow[dr, "c(g_\varphi)"'] & & M(f) \arrow[ll, "h_\varphi"'] \arrow[rr, "h_\psi"] \arrow[dl, "r_\varphi"] \arrow[dr, "r_\psi"'] & & M(g_\psi) \arrow[dl, "c(g_\psi)"] \\
& Z(\varphi) \arrow[rr, "r_{\varphi\psi}"] & & Z(\psi) &
\end{tikzcd}$$

Since $Z(\varphi)$ and $Z(\psi)$, being homeomorphic to $M(f, L \times Q)$ and $M(f, K \times Q)$ respectively, stabilize to Hilbert cube manifolds by Lemma 2, and since $c(g_\varphi)$ and $c(g_\psi)$ stabilize to near-homeomorphisms, again by Lemma 2, the maps r_φ and r_ψ stabilize to near-homeomorphisms. Therefore, by [3, Lemma 5.1] (which states that if X, Y and Z are compact metric spaces, $\alpha : X \to Y$ and $\gamma : Y \to Z$ are maps, and α and $\beta = \gamma\alpha$ are near-homeomorphisms, then γ is a near-homeomorphism), $r_{\varphi\psi}$ stabilizes to a near-homeomorphism. □

Lemma 4. *If $f : K \times Q \to Y$ is a map, where K is a finite simplicial complex and Y stabilizes to a compact Hilbert cube manifold, then for any closed subset A of $K \times Q$, $M(f, A)$ stabilizes to a Hilbert cube manifold homeomorphic to $Y \times Q$ and the relative collapse $c(f, A)$ stabilizes to a near-homeomorphism.*

Proof. Let

$$p_n : K \times Q \to K \times I^n = K \times \prod_{i=1}^{n} I_i$$

be the projection, for each n (here $Q = \Pi_{i=1}^{\infty} I_i$). Also, for each n let T_n be a triangulation regarded as a simplicial subdivision of the cell-complex) of $K \times I^n$ such that if $m < n$ then

$$T_n \cap \left(L \times \prod_{i=m+1}^{n} I_i\right)$$

is a subdivision of $L \times \Pi_{i=m+1}^{n} I_i$ for each subcomplex L of T_m. Assume also that the mesh of T_n goes to zero as $n \to \infty$ and that if $N_n(A)$ is the closed simplicial neighborhood of $p_n(A)$ in T_n, i.e., the union of all closed simplices of T_n meeting $p_n(A)$, then $N_{n+1}(A)$ lies in the topological interior of $N_n(A) \times I_{n+1}$. (Note that $A = \bigcap_{n=1}^{\infty} p_n^{-1}(N_n(A))$.)

Now, for any positive number ϵ and any map

$$\varphi_n : K \times Q \to I$$

which factors as $\varphi_n = \psi_n p_n$, where $\psi_n : K \times I^n \to I$ and $\psi_n^{-1}(0) = N_n(A)$, there is another map

$$\varphi_{n+1} : K \times Q \to I$$

of the form $\varphi_{n+1} = \psi_{n+1} p_{n+1}$, where

$$\psi_{n+1} : K \times I^{n+1} \to I, \qquad \psi_{n+1}^{-1}(0) = N_{n+1}(A),$$

$$\tfrac{1}{2}\epsilon > \varphi_{n+1}(x) - \varphi_n(x) \geq 0 \quad \text{for each } x \in Q.$$

Therefore by Lemma 3 (which produces its inverse) there is a homeomorphism

$$h_{n,\epsilon} : Z(\varphi_n) \times Q \to Z(\varphi_{n+1}) \times Q$$

which moves no point as much as ϵ. Just as a sufficiently rapidly converging (e.g., uniformly Cauchy) sequence of mappings from one compact metric space X_1 to another X_2 converges to a mapping, so a sequence h_n of embeddings of X_1 in X_2 will converge to an embedding if it converges rapidly enough (e.g., if it is uniformly Cauchy and

$$\max\{d_2(h_n(x), h_m(x)) \mid x \in X_1,\ m > n\}$$

$$\leq \tfrac{1}{3}\min\{d_2(h_n(x), h_n(x')) \mid x, x' \in X_1,\ d_1(x, x') \geq 1/n\},$$

where d_i is a metric for X_i, $i = 1, 2$.) Hence a sequence $\varphi_n : K \times Q \to I$ of mappings may be chosen inductively such that:

(1) $\varphi_n^{-1}(0) = p_n^{-1}(N_n(A))$;

(2) φ_n converges uniformly to a map $\varphi : K \times Q \to I$ with $\varphi^{-1}(0) = A$;

(3) there is a sequence of homeomorphisms $g_n : M(f) \times Q \to Z(\varphi_n) \times Q$ which converges to a homeomorphism $g : M(f) \times Q \to Z(\varphi) \times Q$. (This may be done by using the fact that r_{φ_1} stabilizes to a near-homeomorphism by Lemma 3 to obtain g_1 and by constructing each g_{n+1} to be $h_{n+1,\epsilon}\, g_n$ for some ϵ dependent on n.)

In fact, for any preassigned positive number δ, the g_n's may be so chosen that g is within δ of $r_\varphi \times 1_Q$. Thus, $Z(\varphi)$, hence $M(f, A)$, stabilizes to a Hilbert cube manifold and r_φ stabilizes to a near-homeomorphism. [3, Lemma 5.1] then guarantees that

$$c(f) \mid Z(\varphi) : Z(\varphi) \to Y$$

stabilizes to a near-homeomorphism, as it is $c(f) r_\varphi^{-1}$; so $c(f, A)$, also stabilizes to a near-homeomorphism because there is a homeomorphism $h : M(f, A) \to Z(\varphi)$ such that $c(f)h = c(f, A)$. □

Lemma 5. *For each compact Hilbert cube manifold X there is a finite open cover $\{U_i\}_{i=1}^n$ of X such that for each i there is a finite simplical pair (K_i, L_i) and a homeomorphism of compact pairs*

$$h_i : (K_i \times Q, L_i \times Q) \to (\overline{U}_i, \overline{U}_i - U_i),$$

where $\overline{U}_i$ denotes the closure of U_i.

Proof. Let $V_1, ..., V_n$ be a finite open cover of X by sets homeomorphic to open subsets of Q, and let $f_i : V_i \to Q$ be an open embedding, $i = 1, ..., n$. Let $\{W_i\}_{i=1}^n$ be an open cover of X such that for each i, $\overline{W}_i \subset V_i$. Now let

$$p_m : Q \to I^m = \prod_{i=1}^{m} I_i$$

be the projection for each m and observe that, by compactness, there is for each i an integer $m(i)$ large enough that

$$p_{m(i)}^{-1}(p_{m(i)} f_i(\overline{W}_i)) \subset f_i(V_i).$$

Then, for a sufficiently fine triangulation of $I^{m(i)}$, there is a closed simplical neighborhood K_i of $p_{m(i)} f_i(\overline{W}_i)$ such that $p_{m(i)}^{-1}(K_i)$ is also in $f_i(V_i)$. Let L_i be the (topological) boundary of K_i in $I^{m(i)}$. The pair

$$p_{m(i)}^{-1}(K_i, L_i) = \left(K_i \times \prod_{j=m(i)+1}^{\infty} I_j, L_i \times \prod_{j=m(i)+1}^{\infty} I_j\right)$$

is naturally homeomorphic to $(K_i \times Q, L_i \times Q)$, and the maps

$$f_i^{-1} \mid p_{m(i)}^{-1}(K_i)$$

define the desired U_i's and h_i's.

4. The relative collapse stabilization theorem

Theorem (Relative Collapse Stabilization). *If $f : X \to Y$ is any map between two spaces which stabilize to compact Hilbert cube manifolds, then for each closed subset A of X the relative mapping cylinder $M(f, A)$ of f reduced modulo A stabilizes to a Hilbert cube manifold homeomorphic to $Y \times Q$; moreover, the relative collapse*

$$c(f, A) : M(f, A) \to Y$$

stabilizes to a near-homeomorphism, as does the reduction map

$$\pi(f, A) : M(f) \to M(f, A).$$

Proof. There are natural homeomorphisms h of $M(f) \times Q$ and $M(f, A) \times Q$ to $M(f \times 1_Q)$ and $M(f \times 1_Q, A \times Q)$, respectively, such that

$$\pi(f, A) \times 1_Q = h^{-1}\, \pi(f \times 1_Q, A \times Q)h,$$

$$c(f, A) \times 1_Q = h^{-1} c(f \times 1_Q, A \times Q)h;$$

therefore, it is sufficient to consider the case in which X is itself a compact Hilbert cube manifold. Assuming this, let $\{U_i\}_{i=1}^n$ be an open cover of X such that for each i there is a finite simplical pair (K_i, L_i) and a homeomorphism of compact pairs

$$h_i : (K_i \times Q, L_i \times Q) \to (\bar{U}_i, \bar{U}_i - U_i)$$

(Lemma 5).

Let $A_1 = h_1^{-1}(A \cap \bar{U}_1) \cup L_1 \times Q$, and let $f_1 = fh_1 : K_1 \times Q \to Y$. By Lemma 4, $M(f_1, A_1)$ stabilizes to a Hilbert cube manifold homeomorphic to $Y \times Q$, and $c(f_1, A_1)$ stabilizes to a near-homeomorphism. Now let

$$A_2 = h_2^{-1}(A \cap \bar{U}_2) \cup L_2 \times Q,$$

and define $f_2 : K_2 \times Q \to M(f_1, A_1)$ by

$$f_2(x) = \begin{cases} fh_2(x) & \text{if } h_2(x) \notin \bar{U}_1; \\ \pi(f_1, A_1)([h_1^{-1} h_2(x), 1]), & \text{if } h_2(x) \in \bar{U}_1. \end{cases}$$

(Here, Y is regarded, as usual, as the base of $M(f_1, A_1)$, so f is regarded as a map from X into $M(f_1, A_1)$.) Again by Lemma 4, $M(f_2, A_2)$ stabilizes to a Hilbert cube manifold homeomorphic to $Y \times Q$, and $c(f_2, A_2)$ stabilizes to a near-homeomorphism.

If $n = 2$, then by using the $Z(\varphi)$ idea introduced before Lemma 3, it is easy to see that there is a homeomorphism h of $M(f_2, A_2)$ onto $Z(\varphi) \subset M(f)$, for some map $\varphi : X \to I$ with $A = \varphi^{-1}(0)$, such that $c(f_1, A_1)\, c(f_2, A_2) = c(f)h$. As $M(f, A)$ is homeomorphic to $Z(\varphi)$, hence $M(f_2, A_2)$, it stabilizes to a Hilbert cube manifold homeomorphic to $Y \times Q$, and as $c(f, A)$ is topologically equivalent to $c(f) \mid Z(\varphi)$, it is also topologically equivalent to $c(f_1, A_1)\, c(f_2, A_2)$ and stabilizes to a near-homeomorphism. Thus, the first two assertions of the Theorem are true if $n = 2$.

If $n > 2$, then defining A_i to be $h_i^{-1}(A \cap \bar{U}_i) \cup L_i \times Q$ and $f_i : K_i \times Q \to M(f_{i-1}, A_{i-1})$ by

$$f_i(x) = \begin{cases} fh_i(x), & \text{if } h_i(x) \notin \bigcup_{j<i} U_j; \\ \pi(f_j, A_j)([h_j^{-1} h_i(x), 1]), & \text{if } h_i(x) \in \bar{U}_j \text{ and} \\ & j = \max\{k < i \mid h_i(x) \in \bar{U}_k\}, \end{cases}$$

allows the same argument as that for the case $n = 2$ to be carried out on $M(f_n, A_n)$.

To complete the proof, that is, to show that $\pi(f, A) : M(f) \to M(f, A)$ stabilizes to a near-homeomorphism, let $\varphi : X \to I$ be a map with $A = \varphi^{-1}(0)$, and let $h : M(f, A) \to Z(\varphi) \subset M(f)$ be the homeomorphism defined before Lemma 3 ($h([x, t]) = [x, \varphi(x)t]$.) For any map $\psi : X \to I$ with $A = \psi^{-1}(0)$, $\varphi^{-1}(1) = \psi^{-1}(1)$, and $\psi(x) \geqslant \varphi(x)$ for each $x \in X$, there is a homeomorphism g_ψ of $M(f)$ which is the identity on Y such that $g_\psi(Z(\varphi)) = Z(\psi)$ and $g_\psi(\{[x, t] \mid t \in I\}) = \{[x, t] \mid t \in I\}$ for each $x \in X$. The map $h\pi(f, A)$ may be uniformly approximated by maps of the form $g_\psi^{-1} r_\psi$, and since r_ψ stabilizes to a near-homeomorphism, being topologically equivalent to the collapse $c(k, \psi^{-1}(1))$ for $k : X \to Z_\psi$ defined by

$$k(x) = [x, \psi(x)],$$

$k\pi(f, A)$ stabilizes to a near-homeomorphism and so does $\pi(f, A)$. □

Remarks. In fact, in the above proof n may always be taken to be 2 by a result in [1], moreover Chapman has very recently shown that n may be taken to be 1 [2]. The latter result, however, utilizes an infinite-dimensional version of surgery and handle-straightening which the above proof avoids.

The last paragraph of the above proof is necessitated by the two different ways in which $M(f, A)$ is regarded. Note that it is not possible to employ a version of [3, Lemma 5.1] to show that $\pi(f, A) : M(f) \to M(f, A)$ stabilizes to a near-homeomorphism, knowing that $c(f, A) : M(f, A) \to Y$ and $c(f) : M(f) \to Y$ do.

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